LESSON 12

Read about this PROVISIONAL EDITION in the front matter to this book.
Check the NFB website periodically for updates to this lesson.

Now that you are well along in this introductory course it is a good time to supplement the lesson material by reading about the topic in the Nemeth codebook. Study the examples, follow cross-references, revisit previous subject matter. Become familiar with the arrangement of the codebook and use it as a primary source in the future.

The Nemeth codebook that is currently available applies to a transcription following the rules of *English Braille, American Edition*. The BANA Nemeth Code Technical Committee is updating *The Nemeth Braille Code for Mathematics and Science Notation* to align with the use of Nemeth Code in UEB context. In the interim, supplement the codebook with the paper titled "Guidance for Transcription Using the Nemeth Code within UEB Contexts". The *Guidance* document is available as a free download on the BANA website, www.brailleauthority.org.

MODIFIERS AND MODIFIED EXPRESSIONS

12.1 Definition: A modifier is a symbol or a combination of symbols occurring *directly over* or *directly under* its related symbol or expression. Here are some typical examples.

a question mark over an equals sign	$\overset{?}{=}$	
an underlined digit	$3.1\underline{5}$	
a line over two numerals	$1.\overline{37}$	(signifying a repeating decimal)
an arrow over two letters	$\overleftrightarrow{AB}$	(signifying a line)
a caret over a letter	$\hat{\mathbf{k}}$	(vector notation)

12.2 Construction of Simple Modified Expressions – The Five-Step Rule: An expression modified using the 5-step method is initiated and terminated with special indicators.

Multipurpose Indicator	⠸⠼
Termination Indicator	⠸⠼

The position of the modifier (above or below the expression) is also shown with the use of an indicator.

Directly-Over Indicator	
Directly-Under Indicator	

The process of constructing a modified expression in braille is known as "The Five-Step Rule".

(1) The *multipurpose indicator*  is placed immediately before the expression to be modified.

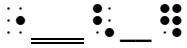
(2) The *expression* to be modified is written second.

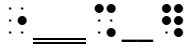
(3) The *directly-over indicator*  or the *directly-under indicator*  is written third to show the position of the modifier.

(4) The *modifying symbol* is written fourth.

(5) The *termination indicator*  is written last to show the end of the modification.

Notice that the multipurpose indicator (dot 5) signals the beginning of the modified expression and the termination indicator signals the completion of the modified expression.

Template for a <u>modifier</u> printed directly above an <u>expression</u>: 

Template for a <u>modifier</u> printed directly below an <u>expression</u>: 

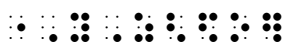
Common Modifiers

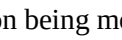
The most commonly-used modifiers are presented in this lesson. You have seen many of these symbols in other contexts in previous lessons. Symbols of the code not shown here may also be used as modifiers.

12.3 Arrows as Modifiers: Any of the arrows of the Nemeth Code may be a modifier. Those used in this section are shown below.

Arrow barbed at right, contracted form	⟶	
Arrow barbed at both ends	⟷	
Arrow shaft with hollow dot at both ends	⟜	
Upper barb only, right-pointing	⇀	
Arrow barbed at right with dashed shaft	--→	

A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is brailled in its contracted form when used as a modifier above or below a math expression.

⋙ $\overrightarrow{\mathrm{YZ}}$ 

"YZ"  is the expression being modified.
It is modified "directly over"  with a right-pointing arrow

⋙ $\underset{\rightarrow}{\mathrm{T}}$ 

"T"  is the expression being modified.
It is modified "directly under" with a right-pointing arrow

The shaft length in print is determined by the width of the expression it modifies. These are not "longer than ordinary" arrow shafts.

⋙ $\overrightarrow{M'N'}$ 

The contracted form of the right-pointing arrow is used in braille.
The arrow shaft covers a wide expression.

All other types of arrows require the appropriate symbol.

⋙ $\overleftrightarrow{\mathrm{AB}}$ 

"AB" is the expression being modified. It is modified "directly over" with an arrow barbed at both ends.

⋙ $\overset{\circ\!-\!\circ}{\mathrm{AB}}$ 

"AB" is the expression being modified. It is modified "directly over" with an arrow with hollow-dotted ends

⋙ $\overrightarrow{\mathbf{OT}}$ 

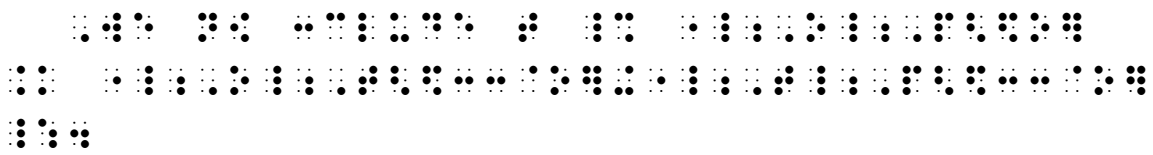
"**OT**" is the expression being modified. It is modified "directly over" with an arrow with an upper right-pointing barb

⋙ $\overset{\dashrightarrow}{\mathrm{OB}}$ 

"OB" is the expression being modified. It is modified "directly over" with a right-pointing arrow with a dashed shaft.

Example 12.3-1 $\overleftrightarrow{\mathrm{PQ}}$ is a line through points P and Q.

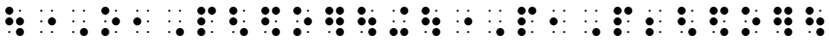
Example 12.3-2 We now conclude that $\overrightarrow{OP} = \overrightarrow{OT} + \overrightarrow{TP}$.



*Reminder from **Lesson 9**: When a linked expression will not fit on one braille line, start a new line with the comparison sign.*

Show exactly what portion of an expression is being modified. In the next example, only the expressions between the grouping symbols are modified. The grouping symbols are brailled outside of the multipurpose and termination indicators.

⋙ $|\overrightarrow{O_1P}| + |\overrightarrow{P_1P_2}|$



"O_1P" and "P_1P_2" are the expressions being modified. Each is modified "directly over" with a right-pointing arrow.

Special Situations Involving Arrows

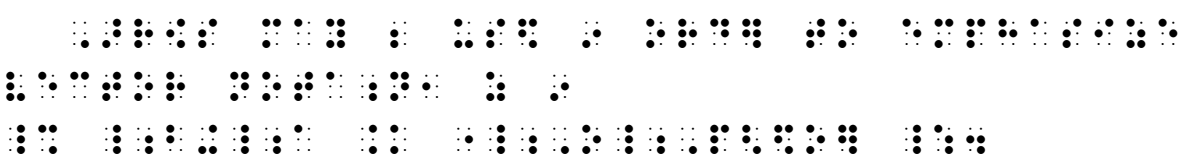
12.3.1 When to Omit Arrows: When *identical arrows* are used above vectors in boldface type *throughout the text*, they are omitted from the braille transcription. (The boldface font attribute of the vectors is considered to be sufficient identification.) If vector arrows are omitted in braille a transcriber's note is required, explaining that the arrows are present in print. For example, on the Transcriber's Notes page the following note should appear: "Arrows printed over vectors are omitted."

⋙ $b + a = \overrightarrow{OP}$ 

The print copy shows vector OP *with a right-pointing arrow above it. If we assume that <u>all</u> vectors in this book are written this way, the boldface is retained in braille but the arrow is omitted. (Note: Only the letters are printed in boldface. The plus sign and the equals sign are in regular type.)*

However, if the author specifically refers to arrows as a notational device, the arrow is brailled.

Example 12.3-3 Arrows may be used in order to emphasize vector notation, as in $b + a = \overrightarrow{OP}$.



12.3.2 When the Arrow is Being Modified: A right-pointing arrow in regular type with a full barb and single shaft of ordinary length is brailled in its uncontracted form when the arrow itself is being modified.

Arrow barbed at right, uncontracted form	⟶	⠸⠒⠒⠒⠸

⋙ $x \xrightarrow{f \circ g} y$ ⠸ ⠸⠒⠸⠒⠒⠸⠒⠸⠒⠸⠒⠒⠸⠒⠸⠒⠸⠸ ⠸

A right-pointing arrow ⠸⠒⠒⠒⠸ is the expression being modified.
It is modified "directly over" ⠸⠒⠸ with the operation "f ∘ g" ⠸⠒⠸⠒⠸

Reminder: In function notation, ∘ is a hollow dot, not the letter "o".

12.3.3 When Other Rules Apply: Some arrangements that may appear to be modified arrows are not brailled using the Five-Step Rule.

Signs of Comparison Compounded Vertically: Horizontal arrows printed one below the other are a sign of comparison compounded vertically. Review this topic in **Lesson 10**.

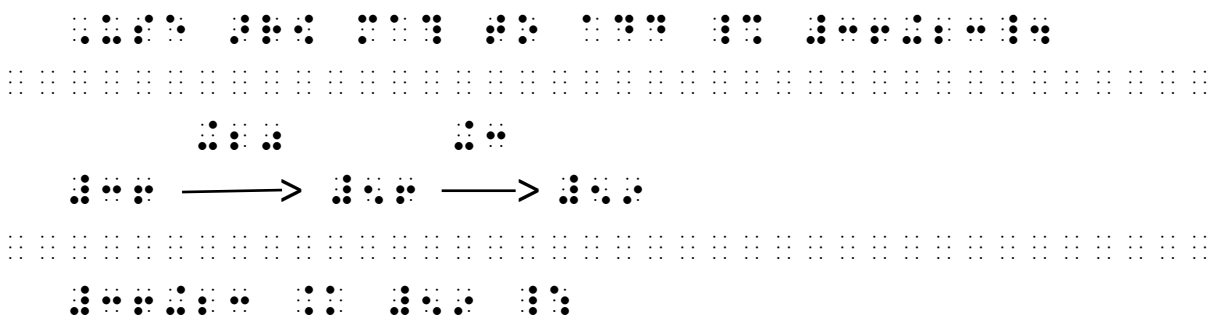
Chemistry: Words or symbols printed above reaction arrows are not brailled as modifiers. Rules for these constructions are found in the *Braille Code for Chemical Notation.*

$$Na_2SO_4 \xrightarrow{H_2O} Na^+ + SO_4^{2-}$$

Elementary Mathematics—"ARROW MATH" in Lower Grades: These arrows are not "modified arrows". They are not being used as a sign of comparison, but rather as a "process". Draw these arrow as a tactile graphic. Refer to BANA's *Guidelines and Standards for Tactile Graphics* for techniques.

Example 12.3-4 Use arrow math to add 36 + 23.

$$36 \xrightarrow{+20} 56 \xrightarrow{+3} 59 \qquad 36 + 23 = 59$$



Instructions: Review arrows in **Lesson 10**. *Reminder*: When the print copy arranges non-spatial itemized material side-by-side across the page and there are no subdivisions, the braille format is changed so that all identifiers start in cell 1.

In the sentence at the end of this practice, assume that all vectors in the document are shown using that particular arrow notation. Show the proper way to omit the vector arrows in the transcription. Include the required transcriber's note after the topic heading. (Review "double-struck" letters in **Lesson 7**.)

PRACTICE 12A

Arrows as Modifiers

1. $\overleftarrow{F}$ 2. $\overleftarrow{AB}$ 3. $\overleftarrow{CD}$ 4. $\overrightarrow{OB} \cup \overrightarrow{OC}$ 5. $\multimap EF$ 6. $GH \multimap\!\bullet$ 7. $\underleftarrow{T}$

8. $\overleftrightarrow{XZ} \parallel \overleftrightarrow{RS}$ 9. $\overleftrightarrow{AB} + \overleftrightarrow{CD}$ 10. $x \overset{g}{\rightarrow} y \overset{f}{\rightarrow} z$

11. $\overrightarrow{OP} = \mathbf{i}x + \mathbf{j}y$

Vector Addition

$\mathbb{R}$ equals $\overrightarrow{OP}$ equals $\overrightarrow{OM}$ plus $\overrightarrow{MC}$ plus $\overrightarrow{CP}$.

12.4 Carets as Modifiers

Caret	∧	
Inverted Caret	∨	

The BANA Nemeth Code Technical Committee is discussing the fact that the symbol for the inverted caret is the same construction as the opening Nemeth Code indicator.

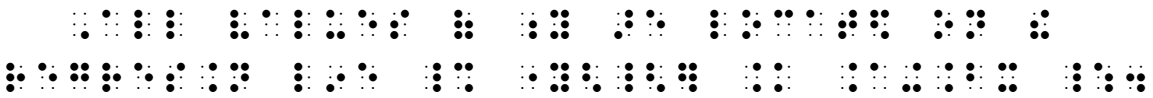
⋙ $\hat{k}$ 

Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

⋙ $x_{\vee}$ 

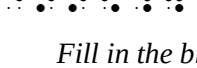
Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

Example 12.4-1 All values of *y* are located on the regression line $\hat{y} = \alpha + \beta x$.



12.5 Horizontal Bar as a Modifier: When more than one digit or letter or expression is modified by a single horizontal bar, the five-step rule is applied. Only one bar symbol is brailled.

Horizontal Bar	—	

⋙ $\overline{99}$ 

Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

Because the numeral is not preceded by a space, a numeric indicator is not used.

The BANA Nemeth Code Technical Committee is discussing details regarding the treatment of underlined numbers. Transcriptions shown below are according to the current Nemeth Code.

⋙ $\underline{419}$

Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

Because the numeral is not preceded by a space, a numeric indicator is not used.

⋙ $\$11.\underline{25}$

Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

The decimal point is not part of the modified expression. The bar is under the digits only.

⋙ $1.\overline{37}$

Fill in the blanks: "___" is the expression being modified. It is modified "directly _____" with a ____.

The decimal point is not part of the modified expression. The bar is over the digits only.

Example 12.5-1

Prove that $\overline{\mathrm{OP}} + \overline{\mathrm{QR}} = \overline{\mathrm{OR}}$.

Example 12.5-2

The fraction $\frac{3}{7}$ expressed as a decimal is .428571428571428571 … . Written as a repeating decimal,

$$\frac{3}{7} = .\overline{428571}\,.$$

The decimal point is not part of the modified expression. The multipurpose indicator is brailled in the cell immediately preceding the numeral "4".

12.5.1 The Contracted Form of Bar Over/Bar Under: When a single digit or a single letter in any typeform or any alphabet is modified only by a single horizontal bar either directly over or directly under it, the five-step rule is not applied. Instead, the contracted form of modified expressions is brailled.

12.5.1.a Bar Above: When a single digit or a single letter (in any typeform or any alphabet) is modified only by a single horizontal bar directly over it, the symbol for the bar is placed immediately after the digit or letter modified.

⋙ $\overline{x}$

⋙ $\overline{3}$

⋙ $\overline{\mathrm{C}}'$

⋙ $\overline{\lambda}_t$

Example 12.5-3 The $\overline{7}'$s are repeated.

The apostrophe-s relates to the numeral, therefore it is brailled inside the switches. The punctuation indicator prevents the apostrophe from being misread as a prime sign.

Example 12.5-4

Arabic	Roman
5,000	$\overline{\mathrm{V}}$
10,000	$\overline{\mathrm{X}}$
50,000	$\overline{\mathrm{L}}$
100,000	$\overline{\mathrm{C}}$

The Arabic numbers in the first column can be brailled in either UEB or Nemeth Code. The Roman numerals in column two require a code switch because they are modified letters. Smoother reading is accomplished by brailling both columns in Nemeth Code.

*(Review simple table format in **Lesson 6**.)*

The BANA Nemeth Code Technical Committee is discussing details regarding the treatment of underlined single numbers and single letters in narrative. Transcriptions shown below are according to the current Nemeth Code.

12.5.1.b Bar Below: When a single digit or a single letter (in any typeform or any alphabet) is modified only by a single horizontal bar directly under it, the directly-under indicator followed by the symbol for the bar is placed immediately after the digit or letter modified.

⋙ $\underline{x}$

⋙ $\underline{\mathrm{R}}^{+}$ 

⋙ $\underline{3}$

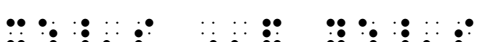
⋙ $5\underline{7}9$ 

More Examples

Study the examples below. Notice when the contracted form of the bar can be used. All simbraille is in Nemeth Code. Switch indicators are omitted.

Examples 12.5-5-11

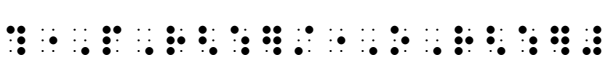
(1) $(\overline{x},\overline{y})$ 

(2) $\overline{x}$'s and $\overline{y}$'s 

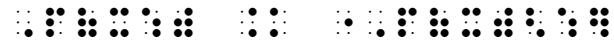
(3) $\overline{x+y}$ 

(4) $\underline{x+y}$ 

(5) $A(\underline{s}n)$

(6) $\frac{\overline{\mathrm{PR}}}{\overline{\mathrm{OR}}}$ 

(This is a fraction.)

(7) $P(\overline{x})=\overline{P(x)}$ 

Instructions: Treat the three examples of unit vectors in problem #2 as displayed mathematical material.

PRACTICE 12B

Carets and Bars as Modifiers

1. Unit vectors can be denoted with normal vector notation, i or $\vec{\imath}$, or with standard unit vector notation, $\hat{\mathbf{i}}$, spoken "i-hat".

2. Here are examples of unit vectors in various coordinate systems.

   Cartesian coordinate system: $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$

   Cylindrical coordinate system: $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$

   Spherical coordinate system: $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$

3. $1.142857\overline{142857}$

4. $\overline{5}$, $\underline{3}$

5. $\frac{7}{15} = .4\overline{6}$

6. $2 \cdot 3 = \overline{2} \cdot 3 = \overline{2 \cdot 3}$

7. $\overline{\mathrm{PQ}}$, $\overline{\mathrm{x}'}$, $\overline{\mathrm{R}''\mathrm{S}''}$

8. $\overline{\mathbf{s}}$, $\overline{\alpha}$, $\overline{m}'$

9. $\mathrm{m}\overline{\mathrm{BC}} = \mathrm{a}$

10. $\overline{\mathrm{C}} = 100 \times 1000$

11. $F = 2\pi\overline{r}l$

12. $\mathrm{g}(\overline{\mathrm{x}}\,\overline{\mathrm{y}}) = \mathrm{g}(\overline{\mathrm{xy}})$

13. $\mathrm{P}(\overline{\mathrm{a} + \mathrm{bi}}) = \overline{0} = 0$

14. $3.141\underline{59}$

15. $\hat{x}_i = 0.5(\overline{x}_i + \underline{x}_i)$

12.6 Other Symbols as Modifiers: Apply the five-step rule for the construction of simple modified expressions when the following symbols are used as modifiers.

12.6.1 Dot

Dot	•	⠨

In print, the recurrence of one digit in a decimal numeral may be indicated by one dot over the recurring digit.

⋙ $\dot{.4}$ ⠨⠲⠼⠙

"4" is the expression being modified. It is modified "directly over" with a single dot.

⋙ $\frac{1}{6} = 0.166\dot{6}\ldots$ ⠼⠁⠌⠼⠋ ⠨⠅ ⠼⠴⠲⠼⠁⠋⠋⠨⠼⠋ ⠄⠄⠄

The last "6" is the expression being modified. It is modified "directly over" with a single dot.

The recurrence of one or more digits in a decimal numeral may be indicated by one dot over each recurring digit. In braille *only one dot* is used as a modifier. The dot is placed after the last modified digit in the recurring sequence.

⋙ $1.\dot{3}7\dot{5}$ ⠼⠁⠲⠨⠼⠉⠑⠨⠼⠑

In print, a single dot is shown over each of the three digits 3 7 and 5. In braille, only one dot is used as a modifier. The multipurpose indicator and termination indicator clearly show what is included within the modified expression.

Example 12.6-1 Expressed as a fraction, $0.90\dot{9}0\dot{9} = \frac{1}{11}$.

⠠⠲⠼⠊⠚⠨⠼⠊⠚⠨⠼⠊ ⠠⠶ ⠼⠁⠌⠼⠁⠁ ⠠⠲
⠼⠴⠲⠼⠚⠋⠨⠼⠚⠨⠼⠊⠋⠨⠼⠊ ⠼⠁⠌⠼⠁⠁ ⠠⠲

In print, a single dot is shown over the last digit 0 and a single dot is shown over the last digit 9.

When one or more dots occur over or under a single letter or numeral, the symbol for the dot is used as many times as necessary to conform with the print text.

⋙ $\dddot{x}$ ⠨⠨⠭

⋙ $\underset{\cdot\cdot}{u}$ ⠨⠥

Example 12.6-2 Prove $\ddot{x} = \frac{d^2x}{dt^2}$

⠠⠏⠗⠕⠧⠑ ⠠⠶ ⠙⠌⠭⠘⠆ ⠌ ⠙⠞⠘⠆

12.6.2 Arc

Arc Concave Upward	⌒	⠐⠉

Example 12.6-3 Consider $\angle ABD = \frac{1}{2}\overset{\frown}{DC}$.

⠐⠉⠐⠣⠠⠙⠠⠃⠠⠁⠐⠜⠀⠐⠜⠐⠉⠠⠙⠠⠉⠐⠜⠀⠐⠣⠐⠉⠐⠜⠐⠣⠠⠙⠠⠉⠐⠜⠐⠜

12.6.3 Tilde

Single Tilde	~	⠈⠔
Extended Tilde	~~	⠈⠔⠈⠔

⋙ $\tilde{u} = 0.8$ ⠈⠔⠣⠥⠻⠀⠀⠨⠅⠀⠀⠼⠚⠲⠓

⋙ $\widetilde{A} \cup \widetilde{B}$ ⠈⠔⠣⠠⠁⠻⠐⠬⠈⠔⠣⠠⠃⠻

⋙ $\overset{\sim}{r+s+t}$ ⠈⠔⠣⠗⠬⠎⠬⠞⠻

Remember, an extended tilde has more than one peak. The next example is a single tilde even though it is covering several items.

⋙ $\widetilde{r+s}$ ⠈⠔⠣⠗⠬⠎⠻

12.6.4 Question Mark: When the question mark is not functioning as a sign of omission, the punctuation mark is brailled. In mathematical context, a punctuation indicator precedes the question mark to prevent it from being misread as the numeral 8.

Question Mark	**?**	⠐⠢

⋙ $\overset{?}{=}$ ⠨⠅⠐⠢⠨⠅

" = " ⠨⠅ is the expression being modified. It is modified "directly over" ⠐ with a question mark ⠐⠢

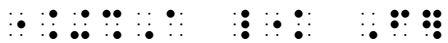
Example 12.6-4 Today our math teacher introduced us to the paradox $0.\overline{9} \overset{?}{=} 1$.

⠠⠞⠙⠁⠽⠀⠍⠁⠞⠓⠀⠞⠑⠁⠉⠓⠻⠀⠊⠝⠞⠗⠕⠙⠥⠉⠻⠀⠥⠎⠀⠞⠕⠀⠞⠓⠑⠀⠏⠁⠗⠁⠙⠕⠭⠀⠼⠚⠲⠼⠊⠀⠨⠅⠐⠢⠨⠅⠀⠼⠁⠲

12.7 Expressions as Modifiers: When a modifier is, itself, a mathematical expression, follow Nemeth Code rules regarding spacing and use of indicators. Each example below contains a comparison sign in the modifier: an arrow, the "inclusion" symbol, a "less than" sign, and an "equals" sign. A space is required before and after a sign of comparison. English letter indicators are not needed when a letter immediately precedes or follows a sign of comparison or when a letter touches an indicator; a numeric indicator is required when a numeral follows a space. The termination indicator ⠸⠄ marks the end of the modifier.

⋙ $\lim_{x \searrow a}$ 

"lim" ⠨⠑⠍ is modified "directly under" ⠸⠡ with the expression $x \searrow a$ ⠸⠡

⋙ $\underset{A \subset F}{\cup}$ 

Fill in the blanks: "⋃" ___ is modified "directly under" ____ with the expression $A \subset F$ _____

⋙ $\underset{i < j}{\Sigma}$ 

Sigma is modified "directly under" with the inequality $i < j$

⋙ $\underset{k = 1}{\prod}$ 

Fill in the blanks: ____ is modified "directly ____" with the equality _____

Reminder: A numeric indicator is required when a numeral follows a space.

Greek capital letters Sigma and Pi are often encountered in this notation.

Sigma	⠨⠎
Pi	⠨⠏

12.7.1 Binomial Coefficient: A binomial coefficient is written as two expressions, one atop the other, enclosed in parentheses. For example, "n choose q" is written like this:

$\binom{n}{q}$

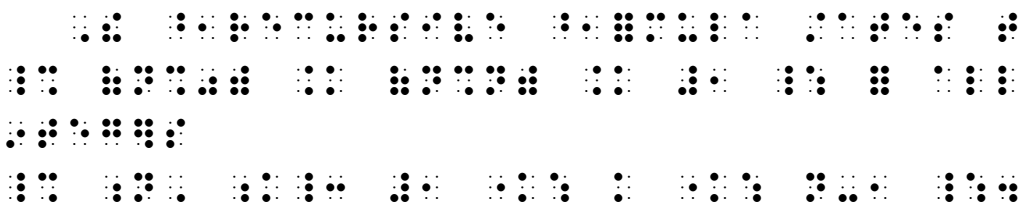
The binomial coefficient does not follow the five-step rule for modified expressions. The opening parenthesis is followed by the upper expression; the directly-under indicator is placed next, followed by the lower expression. The closing parenthesis ends the expression.

Directly-under indicator	⠸⠡

⋙ $\binom{n}{q}$

⩾ $\binom{n-1}{k-1}$

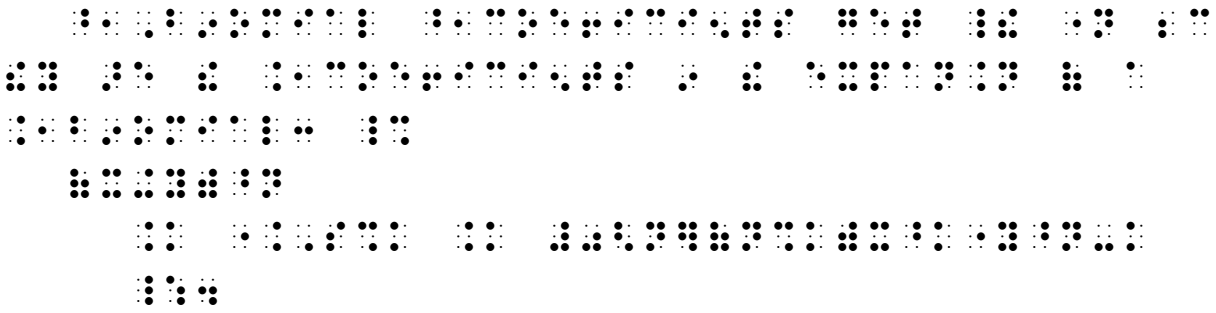
Example 12.7-1 The **recursive formula** states that $\binom{n}{0} = \binom{n}{n} = 1$ for all integers $n, k : 1 \leq k \leq n - 1$.



Reminder from 6.7.16.a – A colon that means "such that" is brailled unspaced from the letter it follows, is preceded by a punctuation indicator, and then is followed by a space. This spacing is applied regardless of the spacing shown in print.

Example 12.7-2 **Binomial coefficients** get their name because they are the *coefficients* in the expansion of a *binomial*:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$



The linked expression is too long to fit on one line. A new line begins with the equals sign placed in the runover position (cell 5 is the runover position for an expression displayed to narrative). The equals sign in the modifier is not a logical division site because it would disrupt the modifier.

PRACTICE 12C

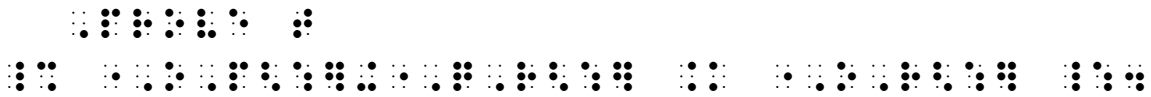
1. In Figure 7.3, if $\overset{\frown}{AB} = \overset{\frown}{CD}$ in circle O, then $\angle AOB = \angle BOC$.
2. $x \stackrel{?}{=} y$ means "does x equal y?"
3. $\sum_{d|n}$ (where d|n means "d divides n").
4. $\binom{t}{p} = R_t^p$
5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for all integers $n \geq 0$.
6. Does $(\tilde{x}, \tilde{y})$ mean $\tilde{x}$ and $\tilde{y}$?
7. More modified expressions:

   a. $.249\dot{9}$ b. $2.431\dot{3}\dot{1}$ c. $\dot{x}\ddot{y} - \dot{y}\ddot{x}$ d. $\underset{\cdot}{a} + \underset{\cdots}{a} = ?$

   e. $f \to \tilde{f}$ f. $\widetilde{x + y}$ g. $\overbrace{(A \cap B \cap C)}$

12.8 Spacing with Modified Expressions: The spacing before and after an entire modified expression is subject to the spacing rules for the symbols preceding or following it.

Example 12.8-1 Prove that $\overline{\mathrm{OP}} + \overline{\mathrm{QR}} = \overline{\mathrm{OR}}$.



There is no space before or after the operation (plus) sign. There is a space before and after the comparison (equals) sign.

When the modifier is wider than the modified symbol, the print copy will insert extra space to clarify what exactly is being modified. In the next example, only the Sigma is modified, not the letter *a*. The space between the Sigma and the *a* clarifies the extent of the modifier, i < j. In braille, however, indicators define the extent of a modifier. There is no need for the space in braille.

$\Sigma_{\mathrm{i<j}}\, a_{ij}$ 

Sigma is modified "directly under" with the inequality $\mathrm{i} < \mathrm{j}$.
The termination indicator signals the completion of the modifier.

PRACTICE 12D

Spacing with Modified Expressions

A. The probability of the event A, written P(A), is defined as

$$\mathrm{P(A)} = \sum_{\wedge} \mathrm{f(x)}$$

where $\sum_{\wedge} \mathrm{f(x)}$ means sum f(x) over those values x_i that are in A.

B. $\underset{\alpha \in A}{\times} \mathrm{A}_{\alpha}$

C. $\underset{i>j}{\prod} (x_i - x_j)$

D. $\overline{7} + 2 \stackrel{?}{=} \overline{7 + 2}$

Modified Expressions and Superscripts / Subscripts

12.9 Modified Expression on the Baseline: A modifier and its related expression must be placed on the same level of writing.

12.9.1 Superscript/Subscript After the Modification: When a modifier affects only the expression and not the superscript or subscript, the modified expression terminates before the superscript (or subscript) is brailled.

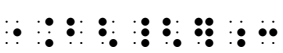
⋙ $\overline{P_1Q}^{\,2} + \overline{QP_2}^{\,2}$ 

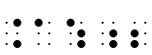
A termination indicator precedes each superscript.

⋙ $\overline{x}^{\,2}$ 

The contracted form of "bar over" does not have an explicit termination indicator. The superscript follows the "bar" symbol, unspaced.

Note that, when a numeric subscript follows a modified letter, a subscript indicator is required. This rule applies whether the letter is modified by the five-step rule or by the contracted form.

⋙ $\hat{\beta}_3$ 

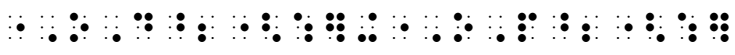
⋙ $\overline{\alpha}_2$ 

⋙ $\underline{\mathrm{N}}_1$ 

12.9.2 Superscript/Subscript Within the Modification: When a modifier affects both an expression on the baseline of writing and its superscript or subscript, the baseline indicator is placed *before* the directly-over (or the directly-under) indicator. This assures that the expression as a whole appears on the same level of writing.

⋙ $\overline{x^2}$ 

The five-step method is used because the modified expression (x^2) is not a single letter. The first dot 5 begins the modified expression. The second dot 5 is a baseline indicator following the superscript "2".

⋙ $\overline{OD^2} + \overline{OP^2}$ 

The first dot 5 begins the first modified expression. The second dot 5 is a baseline indicator following the first superscript "2". Similarly, the third dot 5 begins the second modified expression and the fourth dot 5 is a baseline indicator following the second superscript.

Note that when a numeric subscript which does not require a subscript indicator is part of the modified expression, the subscript indicator is not used.

⋙ $\overline{x_1}$ 

Review **8.10** regarding nonuse of the subscript level indicator.

12.9.3 Binomial Coefficient: Although a binomial coefficient is not technically a "modified expression," notice how this rule applies.

$$\left(\begin{matrix} a_x \\ b_y \end{matrix}\right)$$

The baseline indicator precedes the "directly under" indicator. This keeps that indicator on the same level of writing as the letters "a" and "b".

PRACTICE 12E

Superscripts and Subscripts

A) $\overline{\mathrm{AB}}^{\,2} + \overline{\mathrm{BC}}^{\,2}$

B) $\overline{\mathrm{A}} = [\,\overline{a}_i]$

C) $\sqrt{\ddot{x}^2 + \ddot{y}^2}$

D) $\underline{\mathrm{Z}}^{\circ}$

E) If $\overline{a}_1 = 72$, find $\overline{a}_7$.

F) Draw $\overrightarrow{\mathrm{P_1P_2}}$ if $\mathrm{P_1}$ is the point (1, 3) and $\mathrm{P_2}$ is the point (2, -1).

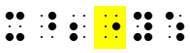
G) $\left(\overline{3^{-1}}\right) \in \mathrm{P}$

H) $\overline{x_1} + \overline{y_1}$

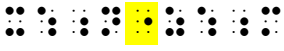
12.9.4 Modified Expression on the Baseline That Follows a Superscript or a Subscript:

When a modified expression written on the baseline of writing immediately follows a superscript or a subscript, several details must be considered in order to determine the necessity of level indicators. Because braille dot 5 has several meanings besides that of baseline indicator (hence the name "multi-purpose indicator") mindful use of that symbol is required in order to give the reader the correct information. Several examples will illustrate. Each "dot 5" is highlighted in the examples.

If a level indicator is needed for the superscript or subscript, the baseline indicator is brailled before starting the modified expression.

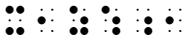
⋙ $x^2\overline{y}$ 

This dot 5 is a baseline indicator, following the superscript 2.
The contracted form of the "bar over" does not use a dot 5.

⋙ $\overline{x}_n\overline{z}_m$ 

This dot 5 is a baseline indicator, following the subscript n.
The contracted form of the "bar over" does not use a dot 5.

If a subscript indicator is not needed, a baseline indicator is not needed before starting the modified expression.

⋙ $x_1\overline{z}_1$ 

Each 1 is printed at the subscript level.

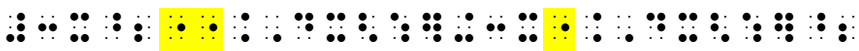
There are no dot 5s in this example. A baseline indicator is not needed when a subscript indicator is not used. (No subscript indicator is needed for the first subscript "1" because it is a numeral that is a right-subscript to a letter.) The contracted form of the "bar over" does not use a dot 5.

⋙ $x_1\overline{z_1}$ 

Each 1 is printed at the subscript level.

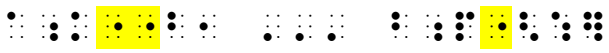
This dot 5 is the start of the modified expression $\overline{z_1}$. A baseline indicator is not needed when a subscript indicator is not used. (No subscript indicator is needed for either subscript because each is a numeral that is a right-subscript to a letter.)

Two indicators may be needed. First, a baseline indicator (dot 5) is used to terminate the effect of the superscript or subscript level indicator. Next, a multipurpose indicator (dot 5) is required to begin the five-step modification.

⋙ $3x^2\overline{\Delta x} + 3x\overline{\Delta x}^{\,2}$


x^2 requires a superscript indicator and the five-step rule is used for the modified expression, so two dot 5s are needed. The baseline indicator is immediately followed by a dot 5 beginning the modified expression $\overline{\Delta x}$.

⋙ $a_k\overline{b_1 \ldots b_p}$



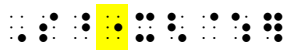
a_k requires a subscript indicator and the five-step rule is used for the modified expression, so two dot 5s are needed. The baseline indicator is immediately followed by a dot 5 beginning the modified expression $\overline{b_1 \ldots b_p}$. Finally, the directly-over indicator is preceded by a baseline indicator to assure that the entire modified expression is brailled on the same level.

⋙ $a_i \prod_{j \neq i} (A - r_j I)$

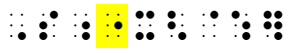
a_i requires a subscript indicator and the five-step rule is used for the modified expression, so two dot 5s are needed. The baseline indicator is immediately followed by a dot 5 beginning the modified expression $\prod_{j \neq i}$

The last dot 5 is a baseline indicator following the subscript "j".

12.10 Modified Expression <u>Within</u> a Superscript or Subscript: Recall that a modifier and its related expression must be placed on the same level of writing. When a modified expression occurs as a superscript or subscript, or as the first part of a superscript or subscript, the level indicator is brailled first, followed by the multipurpose indicator which begins the modified expression. This assures that the expression as a whole appears on the same level of writing.

⋙ $S^{\tilde{x}}$ 

This dot 5 begins the modified expression $\tilde{x}$ (which is in the superscript position). It will not be misread as a baseline indicator because nothing comes between it and the superscript indicator.

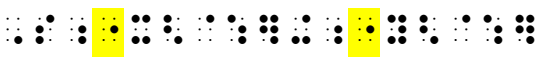
⋙ $S_{\tilde{x}}$ 

This dot 5 begins the modified expression $\tilde{x}$ (which is in the subscript position). It will not be misread as a baseline indicator because nothing comes between it and the subscript indicator.

⋙ ${}_{\tilde{x}}A_1$ 

The first dot 5 begins the modified expression $\tilde{x}$ (which is a left-subscript to the letter A). The second dot 5 is a baseline indicator.

If the modified expression occurs in the middle or at the end of the superscript or subscript, the appropriate level indicator must be repeated before the multipurpose indicator to show continuation of the same level of writing. This assures that the multipurpose indicator will not be misread as a baseline indicator.

⋙ $S_{\tilde{x}+\tilde{y}}$ 

The first dot 5 begins the modified expression $\tilde{x}$. The second dot 5 begins the modified expression $\tilde{y}$. To assure that the second dot 5 is not read as a baseline indicator, the subscript level is restated before the dot 5.

Since the multipurpose indicator is absent in the contracted form of "bar over" or "bar under", the level continues with certainty. It is not necessary to repeat the level indicator when the contracted form is used.

⋙ $e^{a\overline{x}}$ 

PRACTICE 12F

More About Superscripts and Subscripts

(1) $a_0\overline{x}^{\,n} + a_1\overline{x}^{\,n-1}$

(2) $W = \frac{2}{3}\pi r^3 \underline{w}\left(h + \frac{3}{8}r\right)$

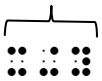
(3) $S^{\tilde{x}}$ and $S^{\tilde{x}+\tilde{y}}$

(4) $D_{\overline{x}}$ or $D_{\overline{x+y}}$

(5) $3_{\overline{x}} - 2_{\overline{x}}$

(6) $_{\overline{n}}A_{\,1}$

12.11 Horizontal Grouping Signs as Modifiers: It is recommended that a horizontal grouping sign be drawn as a tactile graphic as this method more accurately conveys the meaning of the symbol.

⋙ $\overbrace{x+y}$ 

However, a horizontal grouping sign may be represented in braille symbols as a modifier over or under a mathematical expression. In such a case, the entire modified expression is transcribed in accordance with the five-step rule. The opening sign of grouping is used when the modifier is above; the closing sign of grouping is used when the modifier is below.

⋙ $\overbrace{x+y}$  *A brace is printed above* $x+y$.

⋙ $\underbrace{x+y}$  *A brace is printed below* $x+y$.

Modified Signs of Comparison

12.12 Definition: A modified sign of comparison consists of a simple sign of comparison, such as the equals sign or the tilde, modified by a caret, dot, triangle, question mark, vertical bar, or any symbol except another sign of comparison. (When a simple sign of comparison occurs above or below another simple sign of comparison the combination is transcribed as a sign of comparison compounded vertically. See **6.8** for a review of that construction.)

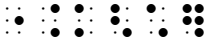
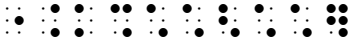
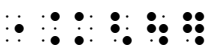
12.13 Transcription: A modified sign of comparison as defined above is transcribed in accordance with the five-step rule for modified expressions.

In addition to the caret and inverted caret seen earlier in this lesson, you may also encounter a left- or right-pointing caret in a modified sign of comparison. Do not confuse these two symbols with the "less than" and "greater than" comparison signs. Ask an expert if context does not clarify the identity of this symbol.

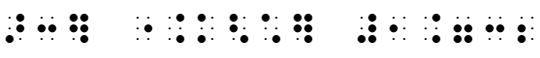
Left-Pointing Caret	<	
Right-Pointing Caret	>	

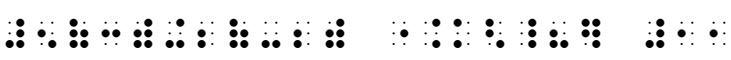
The following list contains the modified equals signs most commonly used.

Modified Equals Sign

Caret Over		
Caret Under ("is projective to")		
Inverted Caret Over		
Left-Pointing Caret Over		
Right-Pointing Caret Over		
Dot Over ("is approximately equal to")		
Dot Over and Dot Under		
Two Dots Over and Two Dots Under		
Hollow Dot Over ("is equal in degrees to")		
Equilateral Triangle Over		
Question Mark Over		
Question Mark Under		
Vertical Bar Over		

We apologize for the blurry images in this lesson. We hope to provide better graphics in future editions.

⋙ $\sqrt{3} \doteq 1.732$ 

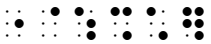
⋙ $5(3) + 2(-2) \stackrel{?}{=} 11$ 

If a symbol is encountered which does not appear on this list, construct a symbol in accordance with the same principles. A created symbol should be listed on the Special Symbols page.

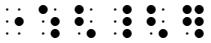
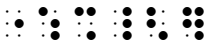
⋙ $A \stackrel{m}{=} B$ 

The remainder of the list provided in the code contains other modified comparison signs most commonly used.

Modified Tilde

Dot Under	$\tilde{\cdot}$	
Dot Over	$\dot{\sim}$	

Modified Horizontal Bar

Caret Over	$\underline{\wedge}$	
Caret Under ("is perspective to")	$\underline{\wedge}$	
Dot under	$\underline{\cdot}$	

Note: If the horizontal bar is modified by a dot *over* it, the combination is a modified sign of operation ("minus with dot over" signifying "proper difference"). The five-step rule is not used for this symbol. See **6.4.8.**

Instructions: Use the five-step rule to show the horizontal grouping sign in the last item.

PRACTICE 12G

Modified Signs of Comparison and More

1. $A \doteq 3.14r^2$
2. $\angle b \triangleq \frac{1}{2}\overset{\frown}{\mathrm{EB}}$
3. The symbol $\triangleq$ is used to make a definition.
4. $x \dot{\sim} \mathcal{N}(0,1)$
5. $x^n = \underbrace{x \cdot x \cdot x \cdot \ldots \cdot x}$

Expressions with More Than One Modifier

12.14 Modifiers of Higher Order: When two or more modifiers occur one above the other and apply *to exactly the same expression*, the second, third, etc. modifiers are "modifiers of higher order." In such cases, the directly-over indicator or the directly-under indicator is doubled, tripled, etc., before each modifier to indicate its position. The termination indicator is used only after the last modifier shown.

Directly-over indicators	
Second order	⠘⠘
Third order	⠘⠘⠘
Directly-under indicators	
Second order	⠨⠨
Third order	⠨⠨⠨
Termination indicator	⠸⠸

$$\overline{\overset{a=3}{x+y}}$$ ⠘⠘⠶⠀⠣⠭⠀⠖⠀⠽⠜⠀⠶⠀⠼⠉

Analysis: The expression "x + y" is modified "directly over" ⠘ with a horizontal bar which, in turn, is modified "directly over" ⠘⠘ with the expression "a = 3".

$$\underline{\overset{a=3}{\underset{b=2}{x+y}}}$$ ⠨⠨⠶⠀⠣⠭⠀⠖⠀⠽⠜⠀⠶⠀⠼⠉⠀⠨⠶⠀⠼⠃

Analysis: The expression "x + y" is modified "directly under" ⠨ with a horizontal bar which, in turn, is modified "directly under" ⠨⠨ with the expression "a = 3" which is modified "directly under" ⠨⠨⠨ with the expression "b = 2".

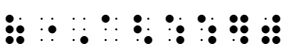
$$\underset{\substack{i,\ j,\ k\\ i<j<k}}{\Sigma}$$ ⠨⠨⠶⠀⠠⠎⠀⠨⠶⠀⠣⠊⠀⠂⠀⠚⠀⠂⠀⠅⠜⠀⠨⠶⠀⠣⠊⠀⠣⠀⠚⠀⠣⠀⠅⠜

Analysis: The Sigma is modified "directly under" ⠨ with "i, j, k" which, in turn, is modified "directly under" ⠨⠨ with the expression "i < j < k".

Review the rules regarding use of the English letter indicator in **Lesson 4**. An English letter indicator is needed for the first letter "j" because it is a "single letter" according to the definition in the Nemeth Code. The other letters are either touching indicators or are preceded or followed by a sign of comparison, so an English letter indicator is not needed.

12.14.1 Parallel Horizontal Bars: When two or more parallel horizontal bars *are the same length and apply to exactly the same expression*, they are treated as a single modifier. In such cases, the directly-over or the directly-under indicator is used only once, and the symbol for the bar is used as many times as necessary to correspond to the print text.

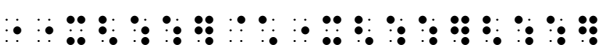
⋙ $\overline{\overline{x}}$ 

⋙ $(\overline{\overline{\mathrm{A}}})$ 

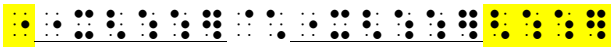
⋙ $\underline{\underline{x}}$ 

Attention: Two parallel horizontal bars must not be misinterpreted as the equals sign, and three parallel horizontal bars must not be misinterpreted as the identity sign. The symbol's meaning can be determined by reading the surrounding text for context. See **6.8** ("Signs of Comparison Compounded Vertically") to review equals sign printed above or below a sign of comparison.

12.15 Individual Modifiers: When two or more modifiers do not apply to exactly the same symbols but *cover different portions of the same expression*, the longer modification encloses the entire modified expression. Within the long expression, each inner expression is modified individually.

⋙ $\overline{\overline{\overline{x} \times \overline{x}}}$ 

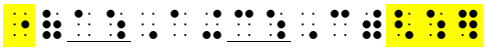
Analysis: In the simulated braille below, the longer modification indicators and symbols are highlighted; the inner modified expressions are underlined.



The five-step rule and the contracted form of transcription may be used together.

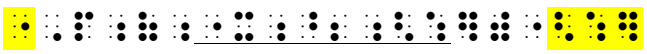
⋙ $\overline{(\overline{\mathrm{a}}\mathrm{A} + \overline{\mathrm{c}}\mathrm{C})}$ 

Analysis: In the simulated braille below, the longer modification indicators and symbols are highlighted; the inner modified expressions are underlined.



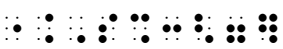
⋙ $\overline{P_{\left(\overline{x^2}\right)}}$ 

Analysis: In the simulated braille below, the longer modification indicators and symbols are highlighted; the inner modified expression is underlined.



Reminder: The subscript level indicator must be repeated before the modified expression indicator so the dot 5 is not misread as a baseline indicator.

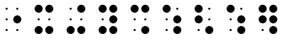
12.16 Simultaneous Modifiers: When an expression is simultaneously modified *both above and below*, the modifier below is brailled first and the modifier above is brailled second. The termination indicator is used only at the end of the entire modification.

⋙ $\sum_{3}^{7}$ 

⋙ $\prod_{k=2}^{6} a_k$ 

Recall from **12.8** that, when the modifier (in this case, k = 2) is wider than the modified symbol (in this case, Pi, Π), the print copy may need to insert extra space to clarify what exactly is being modified. The two letters (Pi and "a") are unspaced in braille because they are part of one math expression.

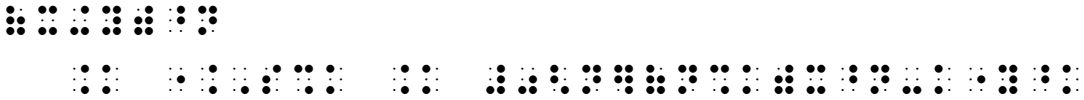
⋙ $\underline{\overline{x}}$ 

⋙ $\underline{\overline{x+y}}$ 

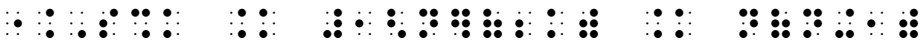
More Examples

As you study these constructions, note the use/nonuse of the English letter indicator and the numeric indicator. All simbraille is in Nemeth Code. Switch indicators are omitted.

Example 12.16-1 $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k$



Example 12.16-2 $\sum_{k=1}^{n} (2k) = n(n+1)$



Example 12.16-3 $\sum_{i,\,j=1}^{n} a_{ij}x_i x_j$

Example 12.16-4 $$\sum_{x_1=0}^{1} \sum_{x_2=0}^{1}$$

⠐⠨⠠⠎⠩⠭⠂⠀⠨⠅⠀⠼⠴⠣⠂⠻⠐⠨⠠⠎⠩⠭⠆⠀⠨⠅⠀⠼⠴⠣⠂⠻

Example 12.16-5 $$\bigcup_{r=1}^{n} \mathrm{A}_r = \mathrm{A}_1 \cup \mathrm{A}_2 \ldots$$

⠐⠨⠬⠩⠗⠀⠨⠅⠀⠼⠂⠣⠝⠻⠠⠁⠰⠗⠀⠨⠅⠀⠠⠁⠂⠨⠬⠠⠁⠆⠀⠄⠄⠄

Example 12.16-6 $$\overline{\overline{\prod_{x \in a} \mathrm{c}'\mathrm{x}}}$$

⠐⠐⠨⠠⠏⠩⠭⠀⠈⠑⠀⠁⠻⠉⠄⠭⠣⠱⠱⠻

Example 12.16-7 $$\underset{a=3}{\underline{\overset{b=2}{\overline{x+y}}}}$$

⠐⠭⠬⠽⠩⠱⠩⠁⠀⠨⠅⠀⠼⠒⠣⠱⠣⠣⠃⠀⠨⠅⠀⠼⠆⠻

Example 12.16-8 $$\sum_{i=1}^{6} \overline{P_{i-1}P_i}$$

⠐⠨⠠⠎⠩⠊⠀⠨⠅⠀⠼⠂⠣⠖⠻⠐⠠⠏⠰⠊⠤⠂⠐⠠⠏⠰⠊⠐⠣⠱⠻

PRACTICE 12H

Expressions with More Than One Modifier

1. $\bar{\bar{9}} \cdot \bar{\bar{3}} = \overline{\overline{9 \cdot 3}}$
2. $\overline{\overline{A^n}}$
3. $\overline{A \cap \overline{\overline{B \cap \bar{\bar{C}}}}}$
4. $\overline{a_{\bar{\bar{n}}} + b_{\underline{p}}}$
5. $\overline{x_{\bar{a}} + y^{\bar{n}}}$
6. $\underline{\bar{\bar{\mathrm{N}}}}$
7. $\sum_{i=1}^{k} \sum_{j=1}^{k}$
8. $\prod_{\substack{j=1 \\ j \neq k}}^{n}$

FORMAL PROOF

12.17 Definition: A proof is a valid argument that establishes the truth of a mathematical statement. A progressive sequence of sentences leads to the conclusion. In a *formal proof,* every step of the argument is shown and each step is supported by a definition or by a previously-proven statement.

12.17.1 Spacing and Margins: A line is left blank before the beginning and after the end of the entire formal proof. The narrative portion of the formal proof begins in cell 3 and its runovers begin in cell 1. If the narrative is in the form of a labeled statement, follow the format directives outlined in **Lesson 11**.

12.17.2 Auxiliary Captions: If the proof contains auxiliary captions such as *Given, Prove,* or *Conclusion*, etc., such captions begin a new paragraph in cell 3 with runovers in cell 1. A line is not skipped above a caption. If a caption is printed in a nonregular typeface, UEB typeform indicators are applied in accordance with the print text. Print capitalization is maintained.

12.17.3 Step-Number Format: In print, a formal proof may be presented in two columns, often headed *Statements* and *Reasons*. The steps may be numbered. The two-column print style of a formal proof is not followed in braille. The following format is used in the transcription.

- A line is left blank before the beginning and after the end of the step-numbered items.
- Each item from the *Reason* column is placed beneath its matching item from the *Statement* column.
- The letter "S" for *Statement* and "R" for *Reason* is placed immediately after the appropriate step number. Similarly, other column headings should be indicated by appropriate letters. The step numbers are brailled in UEB.
- Each step number begins in cell 1 and any runovers begin in cell 3.

A note on the Transcriber's Notes page is included to explain the change in format and to specify the meaning of the letters used to replace the headings. Follow guidelines for the Transcriber's Notes page as presented in *Braille Formats*. For example, "Formal proofs printed in columns headed "Statements" and "Reasons" are brailled as follows: An S or R is added to the step number to show the column in which the step appears. Each step from the Statements column is immediately followed by the corresponding step from the Reasons column."

Example 12.17-1

Theorem 2. All right angles are equal.
Given: ∠ ABC and ∠ DEF are right angles.
Prove: ∠ABC equals ∠DEF.
Proof:

Statements	Reasons
1. ∠ ABC and ∠ DEF are right angles.	1. Given.
2. ∠ ABC = 90°; ∠ DEF = 90°.	2. A right angle contains 90 degrees.
3. ∠ ABC = ∠ DEF.	3. Transitivity postulate.

7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23

Lines 7 and 23: A blank line precedes and follows the proof.

Line 8: When a label is printed in nonregular type (in this example it is printed in boldface) it is transcribed as though it were entirely capitalized and the typeform is ignored. The paragraph style is 3-1. Typeform is also disregarded in the statement when all of the text in the statement is printed in the same nonregular type (in this example, the statement is printed in boldface).

Lines 9-10, 11-12, 13: Each auxiliary caption is italicized (following print typeform), also in 3-1 paragraph style.

Line 14: A blank line precedes the list.

Lines 15-22: Each numbered "S" statement and "R" reason begins in cell 1 with runovers in cell 3.

PRACTICE 12I

Instructions: Braille a Transcriber's Notes page that might appear in a volume which contains the proof shown above. Refer to *Braille Formats* for the structure of a Transcriber's Notes page.

A statement citing the codebook is required in every braille volume that uses the Nemeth Code. In the first paragraph, state the title and edition of the Nemeth codebook as well as any Updates. Something like this:

> Mathematical content is transcribed according to *The Nemeth Braille Code for Mathematics and Science Notation, 1972 Revision,* 2007-2015 Updates including the *Guidance for Transcription Using the Nemeth Code within UEB Contexts.*

In the second paragraph, explain the step-number format as described in **12.17.3**.

Now transcribe this algebraic proof.

PRACTICE 12J

Given: $3x = 7 - \frac{1}{2}x$

Prove: $x = 2$

STEP	REASON
1. $3x = 7 - \frac{1}{2}x$	1. GIVEN
2. $6x = 14 - x$	2. Multiplication Property
3. $7x = 14$	3. Addition Property
4. $x = 2$	4. Division Property

Lesson 10 looked at spatially-arranged addition problems. We now examine multiplication problems, which share some of the same rules.

SPATIAL ARRANGEMENT WITH MULTIPLICATION

The parts of a spatial multiplication problem are labeled below. Note that the multiplication cross is the standard sign used in vertically-arranged multiplication problems.

```
  2     multiplicand
× 3     multiplier
———     separation line
  6     product
```

Review of Format for Spatial Arrangements

1. A blank line is required above and below a spatial arrangement. (10.22)
2. The numeric indicator is not used. (There are some exceptions.) (10.17)
3. An identifier associated with a spatial arrangement is positioned according to rules applied to the specific topic. (10.27)
4. Side-by-side arrangement is allowed according to certain spacing rules. (10.27.1)
5. If a spatial arrangement is brailled on lines 1-2, any symbol of the arrangement cannot fall within three blank spaces of the first symbol of the print page number on line 1. If a spatial arrangement is brailled on lines 24-25, any symbol of the arrangement cannot fall within three blank spaces of the first symbol of the braille page number on line 25. (10.27.2)

12.18 Alignment: In a spatial arrangement for multiplication, the multiplier and multiplicand must be aligned the same way as in the print copy. Any associated symbols such as dollar signs, commas, and decimal points correspond to the print placement.

12.19 Placement of Multiplication Symbol: The multiplication sign must immediately precede the multiplier, regardless of print placement. The multiplication sign is not always present. If there is no symbol, examine the surrounding text to determine that this is indeed a multiplication problem. Then apply alignment rules for multiplication.

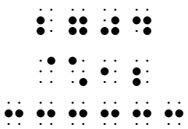
12.20 Separation Line: The separation line extends one cell to the left and to the right of the longest entry appearing above or below it. If there is more than one separation line in a given arrangement/problem, each must be the same length regardless of the way it is printed.

Separation Line (varying in length) ⠒⠒ ⠒⠒ ⠒⠒ ⠒⠒ ⠒⠒ ⠒⠒

Examples Code switch indicators are not shown in the examples in this section.

Example 12.20-1

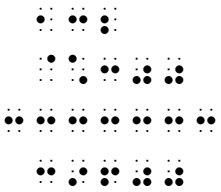
2704
× 12



Alignment: Digits in the multiplier and multiplicand (lines 1 and 2) are vertically aligned the same way as in the print copy. *Placement of Multiplication Symbol:* The multiplication cross is unspaced from the multiplier (line 2) even though this symbol is printed further to the left. *Separation Line:* The separation line extends one cell to the left and to the right of the longest entry appearing above or below it.

Example 12.20-2

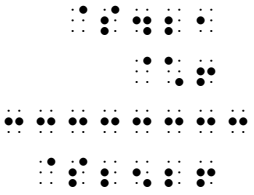
132
× 300
39600



Alignment: Digits in the multiplier and multiplicand (lines 1 and 2) and in the product (line 4) are vertically aligned the same way as in the print copy. *Placement of Multiplication Symbol:* The multiplication cross is unspaced from the multiplier (line 2) even though this symbol is printed further to the left. *Separation Line:* The separation line extends one cell to the left and to the right of the longest entry appearing above or below it.

Example 12.20-3

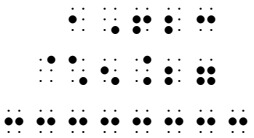
$421
× 6
$2526



Spacing and Alignment: The dollar signs correspond to the print placement.

Example 12.20-4

1,623
× 5.27



Spacing and Alignment: The commas and decimal points correspond to the print placement.

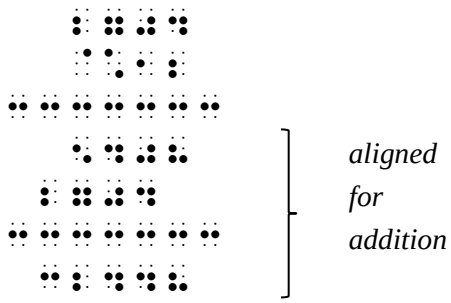
Alignment of Partial Products

12.21 Partial Products: When partial products are shown in a sample problem, note that the final answer (the product) is obtained by *adding* the partial products. Hence, partial products and the final product (the answer) must be aligned for addition.

```
 2704      (multiplicand)
× 12       (multiplier)
 5408      partial product
2704       partial product      } an addition problem
32448      product
```

Example 12.21-1

```
 2704
× 12
 5408
2704
32448
```



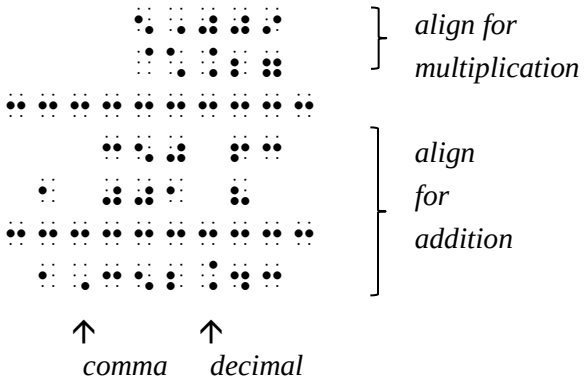
Alignment of Partial Products and Final Product: Lines 4 and 5 constitute the partial products. Alignment follows print. The final product (line 7) is aligned by place value according to the rules of addition.

Separation Lines: In braille, all separation lines are the same length and in the same cells even though they may not appear this way in print.

12.21.1 Spacing: If the product contains a comma or a decimal point, the corresponding cells in the partial products above it are left blank. No blank cells are inserted in the separation lines.

Example 12.21-2

```
 5,009
 × .27
 35063
 10018
1,352.43
```



Instructions: Use side-by-side layout, leaving one blank space between the end of one separation line and the beginning of the next.. Review this format in **Lesson 10.** Include Nemeth Code switch indicators in your transcription.

PRACTICE 12K

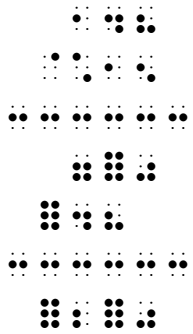
$98	$33	12.12	7,165
× 100	× 200	× 15.3	× 85
$98	$33	12.12	7,165
× 100	× 200	× 15.3	× 85
$9,800	$6,600	3636	35 825
		6060	573 20
		1212	609,025
		185.436	

12.22 Omissions: In order to maintain necessary alignment when an arrangement contains omissions, the long dash, the ellipsis, or shape indicators are not used. *Only the general omission symbol is used*. The same number of omission symbols as shown in print should be brailled.

General Omission Symbol ⠿

Example 12.22-1

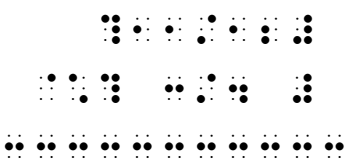
148
× 15
7□0
□48
□2□0



12.23 Fractions and Mixed Numbers: In a multiplication problem which contains fractions and mixed numbers, the terms, indicators, and place values are aligned vertically.

Example 12.23-1

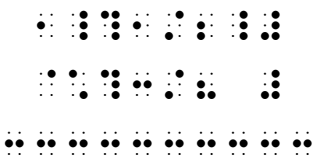
$$\begin{array}{r} \frac{11}{12} \\ \times \frac{3}{4} \\ \hline \end{array}$$



The fraction indicators and the fraction lines are vertically aligned. Numerators and denominators touch the fraction line.

Example 12.23-2

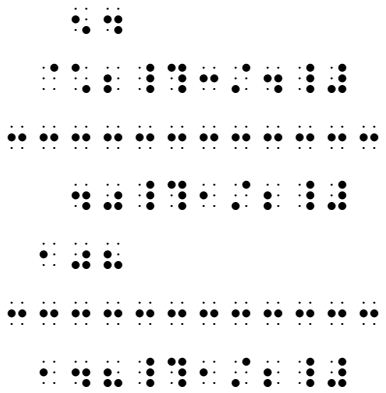
$$\begin{array}{r} 1\frac{1}{2} \\ \times \frac{3}{8} \\ \hline \end{array}$$



The fraction indicators align, including the fractional part of the mixed number. The multiplication cross touches the multiplier.

Example 12.23-3

$$\begin{array}{r} 54 \\ \times 2\frac{3}{4} \\ \hline 40\frac{1}{2} \\ 108 \\ \hline 148\frac{1}{2} \end{array}$$

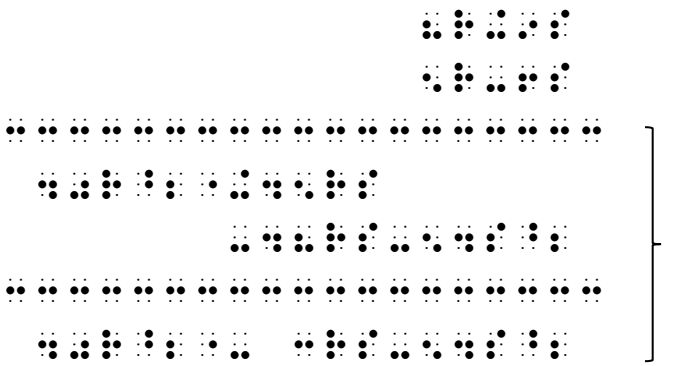


Place values and indicators are aligned throughout.

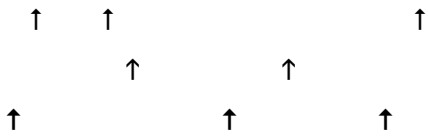
12.24 Polynomials: In a multiplication problem which contains polynomials, the terms and indicators are aligned vertically in the partial products and final product. Above the first separation line, the multiplicand and multiplier are aligned as in the print copy.

Example 12.24-1

$$\begin{array}{r} 8r+9s \\ 5r-6s \\ \hline 40r^2+45rs \\ \quad -48rs-54s^2 \\ \hline 40r^2- 3rs-54s^2 \end{array}$$



aligned for addition



... *braille indicators are aligned*

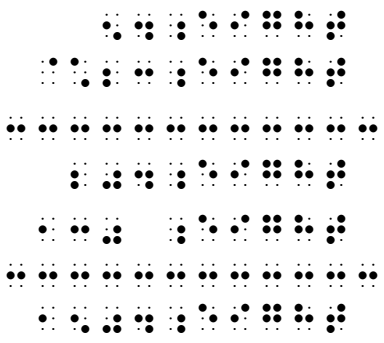
... *operation signs are aligned as in print*

... *terms (r, rs, and s) are aligned as in print*

12.25 Subscripts Denoting Nondecimal Bases: In arrangements which show multiplication of non-decimal bases, the subscript indicators are vertically aligned. The rightmost partial product sets the location of this alignment. This may differ from the spacing shown in the print copy.

Example 12.25-1

$$\begin{array}{r} 54_{\text{eight}} \\ \times\ 23_{\text{eight}} \\ \hline 204_{\text{eight}} \\ 130_{\text{eight}} \\ \hline 1504_{\text{eight}} \end{array}$$



← *rightmost partial product*

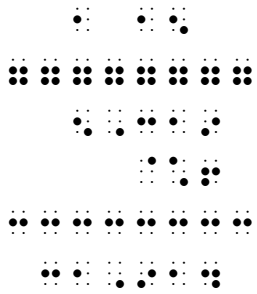
↑ *subscript indicators aligned*

12.26 Carried Numbers with Multiplication: If carried numbers are shown, follow the rules for carried numbers with addition. (**10.28**) A line of carried number indicators the same length as the separation line is inserted between the carried numbers and the first line of the multiplication problem. A carried number line is brailled whether or not the line appears in the print copy. Carried numbers should appear in the same columnar position as in print.

Carried Number Indicator (varying in length)	⠐⠐⠐⠐⠐⠐

Example 12.26-1

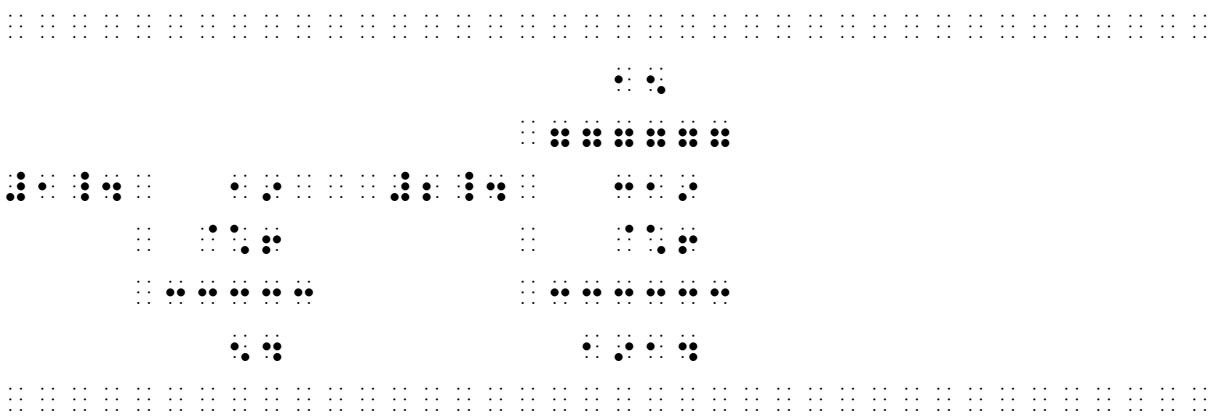
```
  1 15
 5,319
   × 6
------
31,914
```



12.27 Placement of Identifiers with Spatial Multiplication: An identifier, if present, is placed on the first line of the multiplication problem (the multiplicand) regardless of its placement in print. If there are carried numbers, the identifier is still placed on the line with the multiplicand.

Example 12.27-1

```
1.  19        2.  15
                  319
   ×6             ×6
   54            1914
```



Reminders: One blank space comes between the last symbol in the identifier and the symbol furthest left in the overall arrangement, including separation lines. No symbol of one spatial arrangement or its identifier may be less than three cells distant from any symbol on any line of a neighboring arrangement or its identifier, except at the ends of separation lines.

Instructions: Use side-by-side layout. Be sure that three blank cells come between any symbol on any line of one spatial arrangement and any symbol on any line of a neighboring arrangement (separation lines excluded). Include Nemeth Code switch indicators in your transcription.

PRACTICE 12L

1) $\begin{array}{r} \frac{1}{2} \\ \times \frac{15}{16} \\ \hline \end{array}$

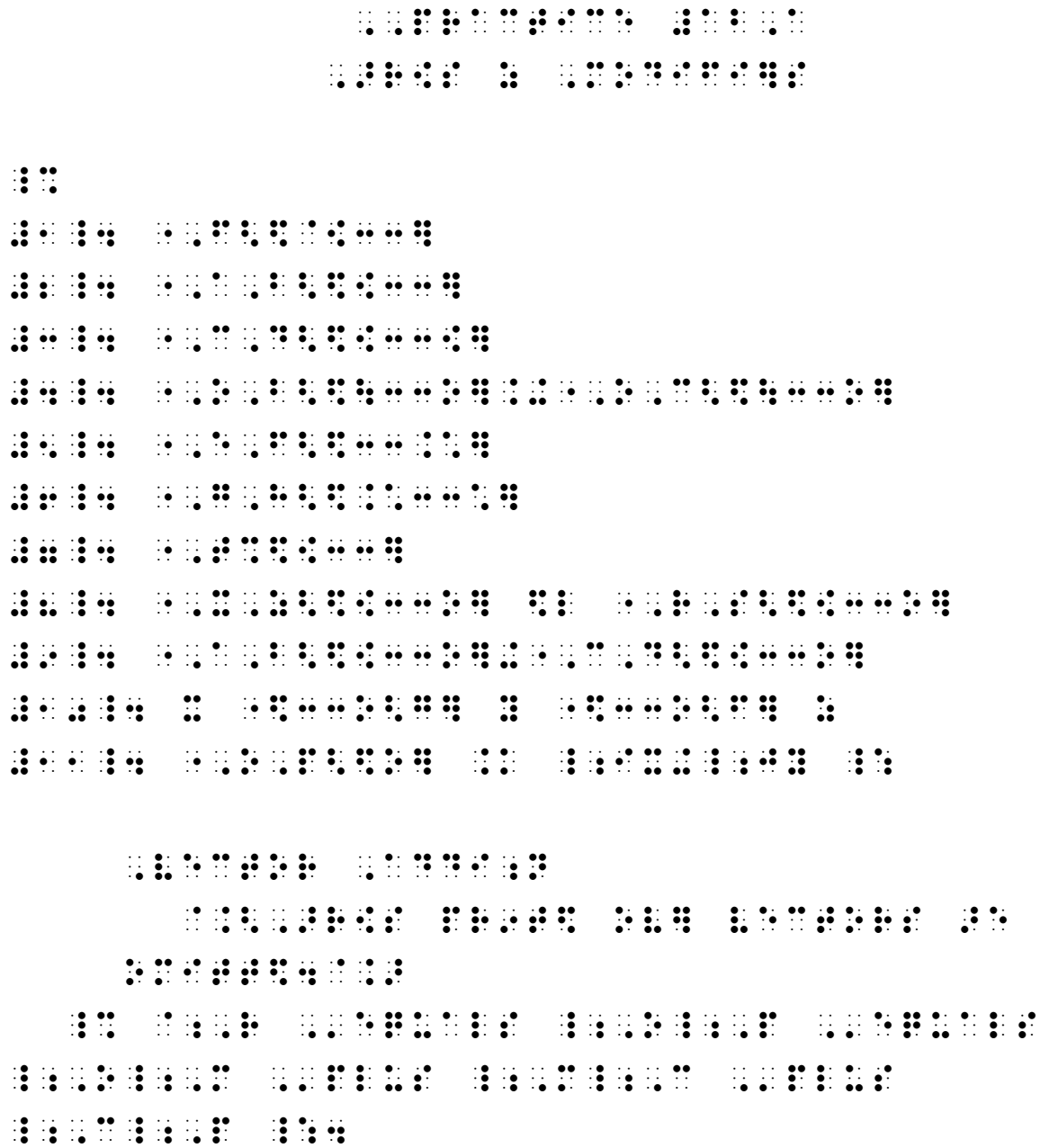
2) $\begin{array}{r} 9\frac{3}{4} \\ \times 4\frac{7}{12} \\ \hline \end{array}$

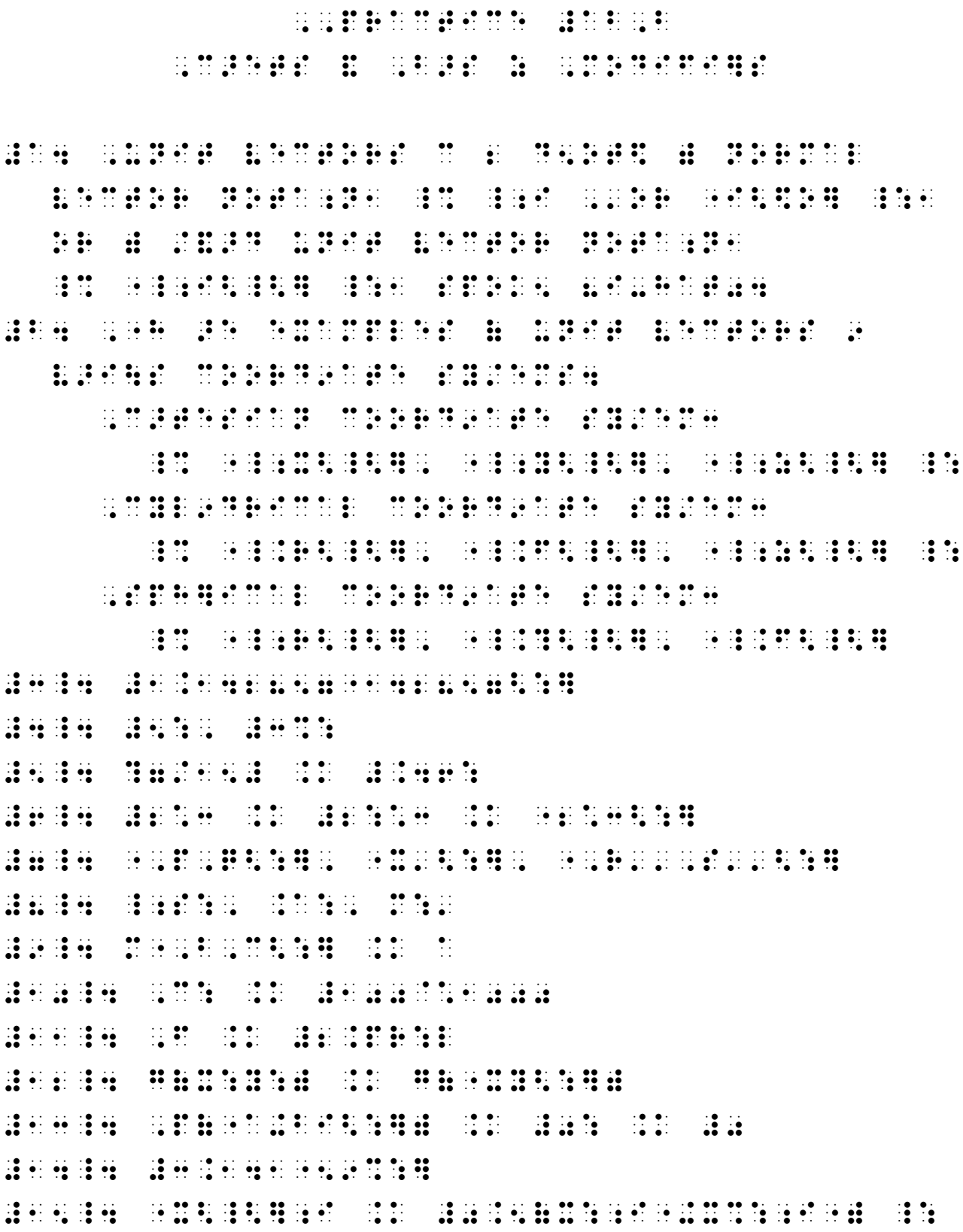
3) $\begin{array}{r} 33 \\ 999 \\ \times\ 4 \\ \hline 3,996 \end{array}$

4) $\begin{array}{r} 3p+6q \\ 11p-2q \\ \hline 33p^2+66pq \\ -\ 6pq-12q^2 \\ \hline 33p^2+60pq-12q^2 \end{array}$

For further practice, see Appendix A—Reading Practice.

ANSWERS TO PRACTICE MATERIAL

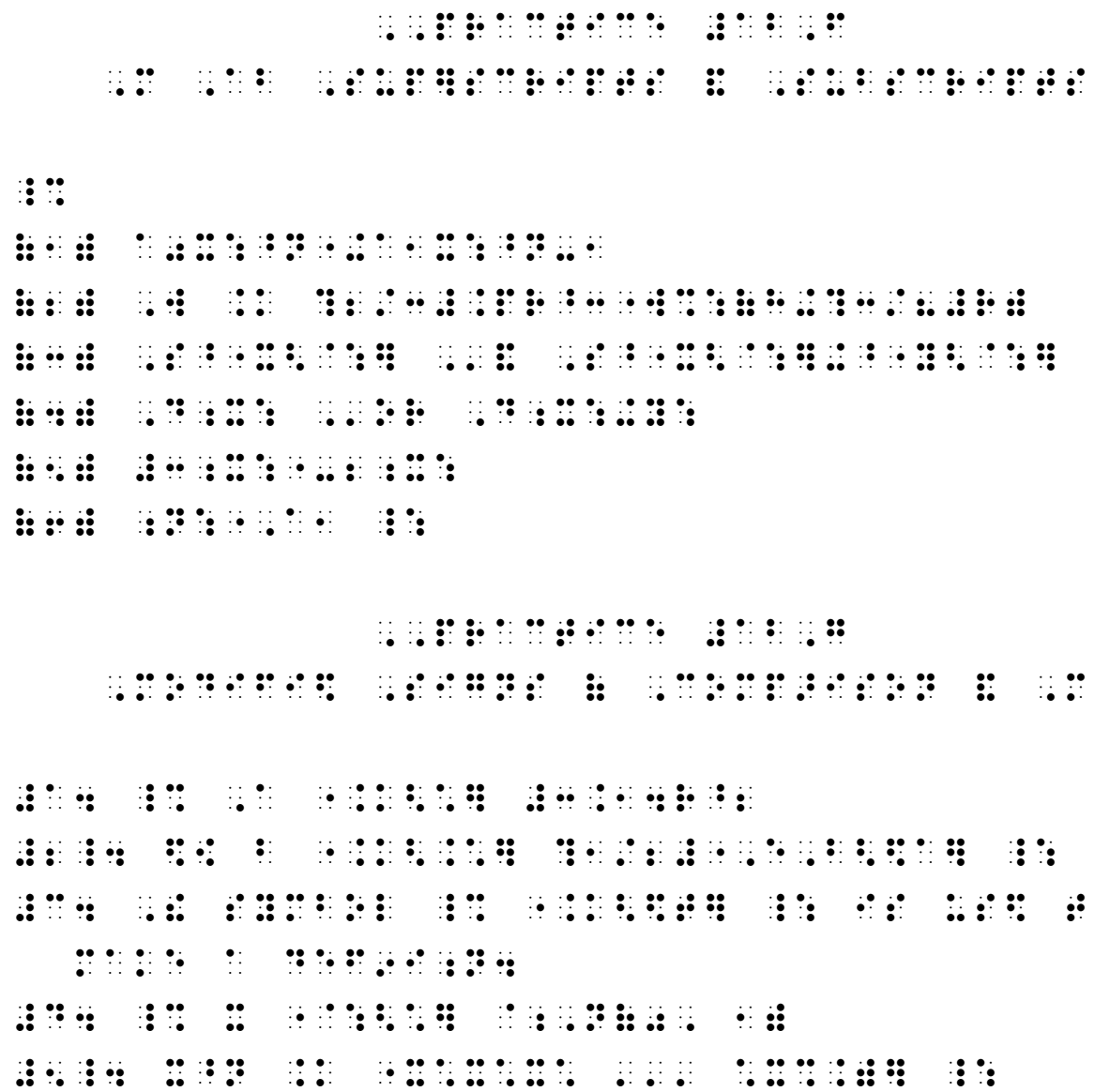




Note: The opening Nemeth Code indicator that starts line 15 may be placed at the end of line 14 instead. This transcription maintains the pattern shown on lines 11 and 13.

⠠⠠⠏⠗⠁⠉⠞⠊⠉⠑ ⠼⠁⠃⠠⠉

⠼⠁⠲ ⠠⠔ ⠠⠋⠊⠛⠥⠗⠑ ⠼⠛⠲⠉⠂ ⠊⠋
    ⠸⠩ ⠐⠠⠁⠠⠃⠣⠫⠁⠻ ⠨⠅ ⠐⠠⠉⠠⠙⠣⠫⠁⠻ ⠸⠱ ⠔ ⠉⠊⠗⠉⠇⠑
    ⠠⠕⠂ ⠮⠝ ⠸⠩ ⠫⠪ ⠠⠁⠠⠕⠠⠃ ⠨⠅ ⠫⠪ ⠠⠃⠠⠕⠠⠉ ⠸⠱⠲
⠼⠃⠲ ⠸⠩ ⠭ ⠐⠨⠅⠣⠸⠦⠻ ⠽ ⠸⠱ ⠍⠂⠝⠎ ⠦⠙⠕⠑⠎ ⠰⠭
    ⠑⠟⠥⠁⠇ ⠰⠽⠦⠴
⠼⠉⠲ ⠸⠩ ⠐⠨⠠⠎⠩⠙⠳⠝⠻ ⠠⠄⠐⠣⠐⠱ ⠙⠳⠝ ⠸⠱ ⠍⠂⠝⠎ ⠦⠰⠙
    ⠙⠊⠧⠊⠙⠑⠎ ⠰⠝⠴⠐⠜⠲
⠼⠙⠲ ⠸⠩ ⠷⠞⠩⠏⠾ ⠨⠅ ⠠⠗⠰⠞⠘⠏
⠼⠢⠸⠲ ⠷⠝⠩⠅⠾ ⠨⠅ ⠷⠝⠤⠂⠩⠅⠤⠂⠾⠬⠷⠝⠤⠂⠩⠅⠾ ⠸⠱ ⠿ ⠁⠇⠇
    ⠔⠞⠑⠛⠻⠎ ⠸⠩ ⠝ ⠨⠂⠱ ⠼⠴⠸⠲
⠼⠖⠸⠲ ⠠⠄⠠⠙⠕⠑⠎ ⠷⠐⠭⠣⠈⠱⠻⠠ ⠐⠽⠣⠈⠱⠻⠾ ⠠⠄⠍⠂⠝
    ⠐⠭⠣⠈⠱⠻ ⠠⠄⠯ ⠐⠽⠣⠈⠱⠻ ⠸⠱⠦
⠼⠛⠲ ⠠⠍ ⠍⠕⠙⠊⠋⠊⠫ ⠑⠭⠏⠗⠑⠎⠨⠝⠎⠒ ⠸⠩
    ⠰⠁⠸⠲ ⠼⠨⠆⠲⠔⠐⠔⠣⠡⠻
    ⠰⠃⠸⠲ ⠼⠆⠨⠲⠒⠂⠐⠒⠂⠣⠡⠻
    ⠰⠉⠸⠲ ⠐⠭⠣⠡⠻⠐⠽⠣⠡⠡⠻⠤⠐⠽⠣⠡⠻⠐⠭⠣⠡⠡⠻
    ⠰⠙⠸⠲ ⠐⠁⠩⠡⠻⠤⠐⠁⠩⠡⠡⠡⠻ ⠨⠅ ⠿
    ⠰⠑⠸⠲ ⠋ ⠫⠕ ⠐⠋⠣⠈⠱⠻
    ⠰⠋⠸⠲ ⠐⠭⠬⠽⠣⠈⠱⠻
    ⠰⠛⠸⠲ ⠷⠐⠠⠁⠨⠩⠠⠃⠨⠩⠠⠉⠣⠈⠱⠻⠾ ⠸⠱



⠠⠠⠏⠗⠁⠉⠞⠊⠉⠑ ⠼⠁⠃⠠⠓

⠠⠑⠭⠏⠗⠑⠎⠨⠝⠎ ⠯ ⠠⠍ ⠠⠹⠁⠝ ⠠⠐⠕ ⠠⠍⠕⠙⠊⠋⠊⠻

⠸⠩

⠼⠂⠸⠲ ⠐⠔⠣⠱⠱⠻⠡⠐⠒⠣⠱⠱⠻ ⠨⠅ ⠐⠔⠡⠒⠣⠱⠱⠻

⠼⠆⠸⠲ ⠐⠠⠁⠘⠝⠐⠣⠱⠱⠻

⠼⠒⠸⠲ ⠐⠠⠁⠨⠩⠐⠠⠃⠨⠩⠠⠉⠱⠣⠱⠻⠣⠱⠻

⠼⠲⠸⠲ ⠐⠁⠰⠝⠱⠐⠬⠃⠰⠏⠱⠐⠣⠱⠻

⠼⠢⠸⠲ ⠐⠭⠰⠁⠱⠐⠬⠽⠘⠝⠱⠐⠣⠱⠻

⠼⠖⠸⠲ ⠐⠠⠝⠩⠱⠣⠱⠱⠻

⠼⠶⠸⠲ ⠐⠨⠠⠎⠩⠊ ⠨⠅ ⠼⠂⠣⠅⠻⠐⠨⠠⠎⠩⠚ ⠨⠅ ⠼⠂⠣⠅⠻

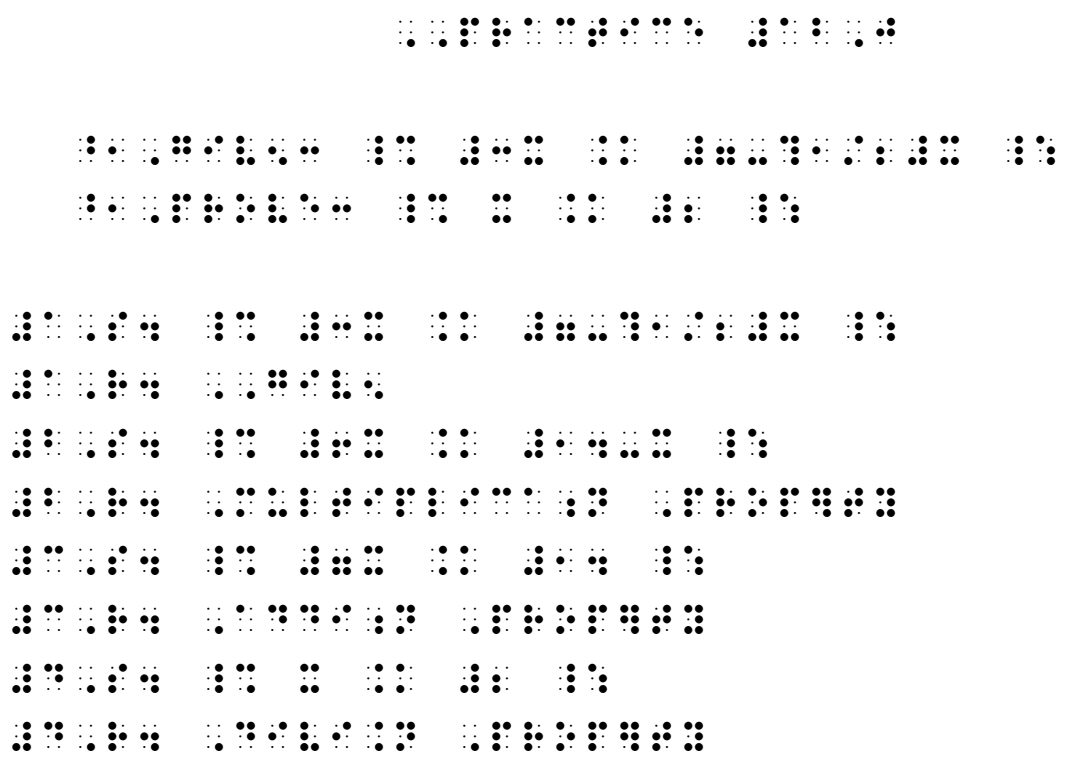
⠼⠦⠸⠲ ⠐⠨⠠⠏⠩⠚ ⠨⠅ ⠼⠂⠩⠩⠚ ⠌⠨⠅ ⠅⠣⠝⠻ ⠸⠱

⠠⠠⠏⠗⠁⠉⠞⠊⠉⠑ ⠼⠁⠃⠠⠊

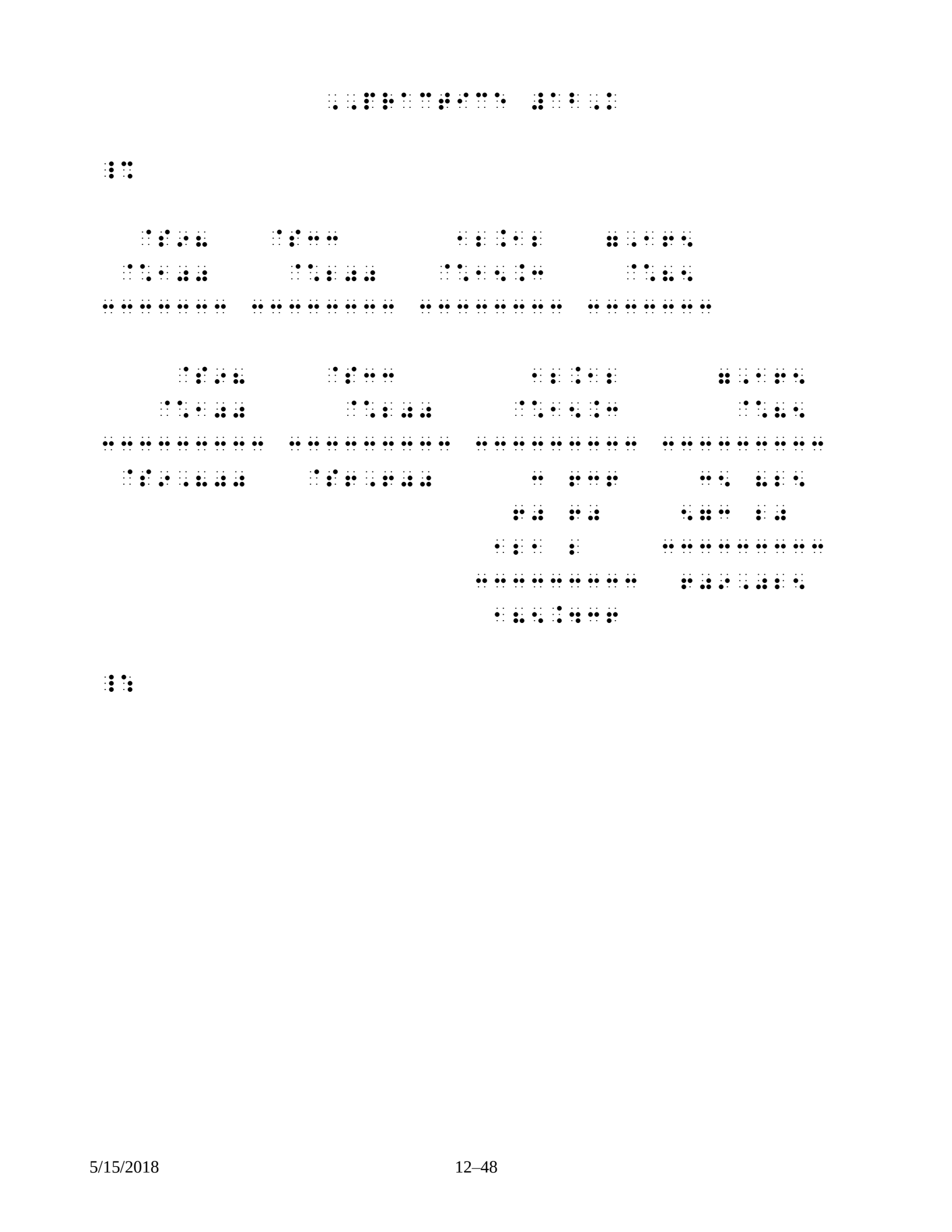
⠠⠠⠞⠗⠁⠝⠎⠉⠗⠊⠃⠻⠄⠠⠎ ⠠⠠⠝⠕⠞⠑⠎

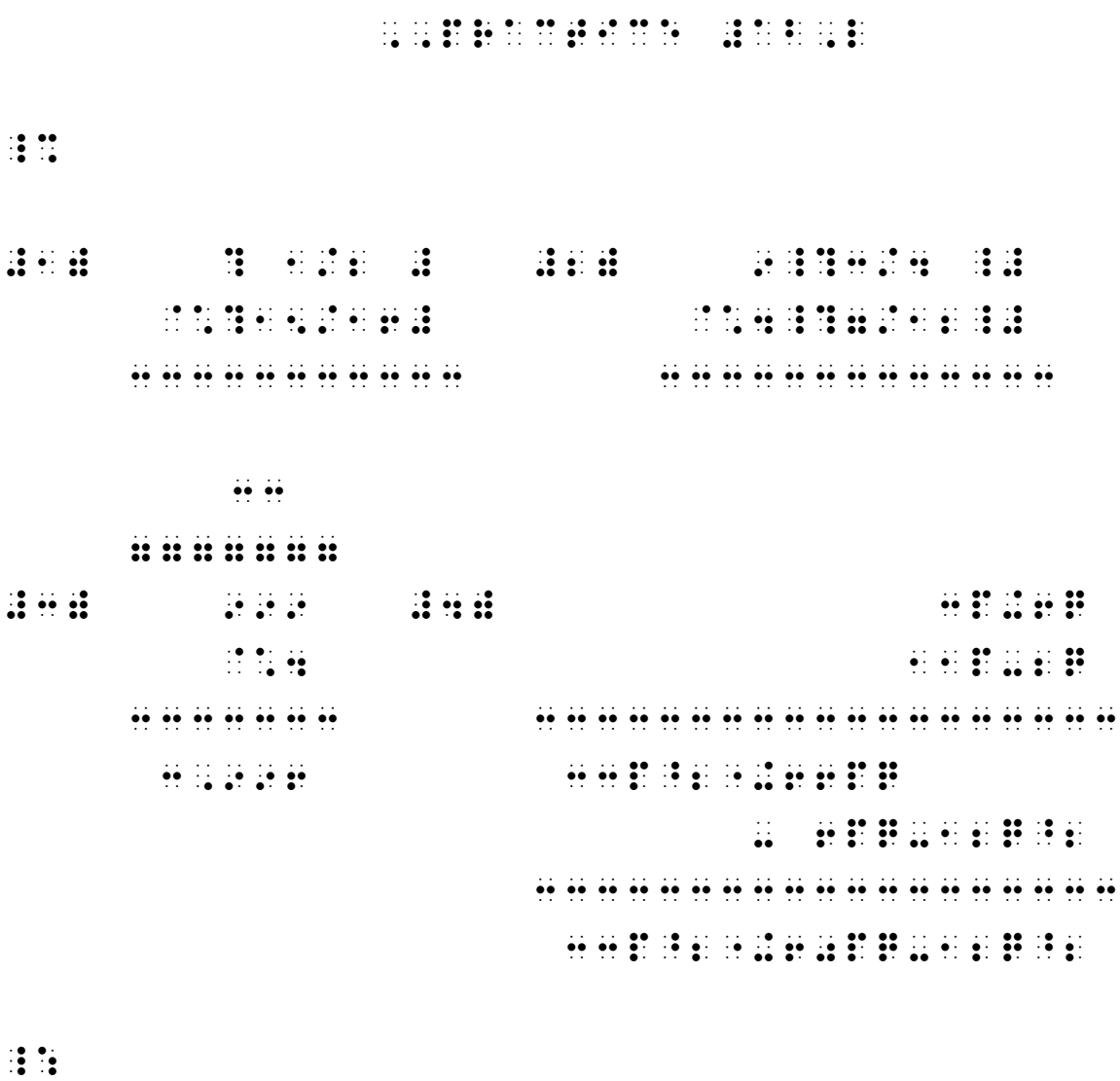
⠠⠍⠁⠮⠍⠁⠞⠊⠉⠁⠇ ⠒⠞⠢⠞ ⠊⠎ ⠞⠗⠁⠝⠎⠉⠗⠊⠃⠫ ⠁⠉ ⠞⠕ ⠨⠶⠠⠮ ⠠⠝⠑⠍⠑⠹ ⠠⠃⠗⠇ ⠠⠉⠕⠙⠑ ⠿ ⠠⠍⠁⠮⠍⠁⠞⠊⠉⠎ ⠯ ⠠⠎⠉⠊⠰⠑ ⠠⠝⠕⠞⠁⠰⠝⠂ ⠼⠁⠊⠛⠃ ⠠⠗⠑⠧⠊⠨⠝⠂ ⠼⠃⠚⠚⠛⠤ ⠼⠃⠚⠁⠑ ⠠⠥⠏⠙⠁⠞⠑⠎ ⠔⠉⠇⠥⠙⠫ ⠮ ⠨⠶⠠⠛⠥⠊⠙⠨⠑ ⠿ ⠠⠞⠗⠁⠝⠎⠉⠗⠊⠏⠰⠝ ⠠⠥⠎⠬ ⠮ ⠠⠝⠑⠍⠑⠹ ⠠⠉⠕⠙⠑ ⠾⠔ ⠠⠠⠥⠑⠃ ⠠⠒⠞⠑⠭⠞⠎⠲⠨⠄

⠠⠿⠍⠁⠇ ⠏⠗⠕⠷⠎ ⠏⠗⠔⠞⠫ ⠔ ⠉⠕⠇⠥⠍⠝⠎ ⠓⠂⠙⠫ ⠦⠠⠌⠁⠞⠑⠰⠞⠎⠴ ⠯ ⠦⠠⠗⠂⠎⠕⠝⠎⠴ ⠞⠑ ⠃⠗⠇⠙ ⠵ ⠋⠕⠇⠇⠪⠎⠒ ⠠⠁⠝ ⠰⠠⠎ ⠕⠗ ⠰⠠⠗ ⠊⠎ ⠁⠙⠙⠫ ⠞⠕ ⠮ ⠌⠑⠏ ⠝⠥⠍⠃⠻ ⠞⠕ ⠩⠪ ⠮ ⠉⠕⠇⠥⠍⠝ ⠔ ⠱ ⠮ ⠌⠑⠏ ⠁⠏⠏⠂⠜⠎⠂ ⠑⠁⠡ ⠌⠑⠏ ⠿ ⠮ ⠠⠌⠁⠞⠑⠰⠞⠎ ⠉⠕⠇⠥⠍⠝ ⠊⠎ ⠊⠍⠍⠇⠽ ⠋⠕⠇⠇⠪⠫ ⠃⠽ ⠮ ⠉⠕⠗⠗⠑⠎⠏⠕⠝⠙⠬ ⠌⠑⠏ ⠋ ⠮ ⠠⠗⠂⠎⠕⠝⠎ ⠉⠕⠇⠥⠍⠝⠲



It is acceptable not to terminate Nemeth Code on line 3, using the single-word switch for ***Prove****. However, all step numbers are to be brailled in UEB.*





EXERCISE 12

Exercise 12 will be available when this course is finished being written and is no longer "Provisional".

Proceed to Lesson 13.